The dynamical equation of the spinning electron

Martín Rivas[*]
Dpto. de Física Teórica,
The University of the Basque Country,
Apdo. 644, 48080 Bilbao, Spain

December 4, 2001

Abstract

By invariance arguments we obtain the relativistic and non-relativistic invariant dynamical equations of the spinning electron. The dynamics can be expressed in terms of the evolution of the point charge which satisfies a fourth order differential equation or, alternatively, by describing the evolution of both the center of mass and center of charge of the particle.

PACS: 03.65.S; 14.60.C
keywords: Semiclassical theories; Electron; Zitterbewegung

[*]e-mail: wtpripem@lg.ehu.es

1 Introduction

The latest LEP experiments at CERN suggest that the electron charge is confined within a region of radius $R_e < 10^{-19}$m. Nevertheless, the quantum mechanical behaviour of the electron appears at distances of the order of its Compton's wave length $\lambda_C = \hbar/mc \simeq 10^{-13}$m, which is six orders of magnitude larger.

One possibility to reconcile these features is the assumption from the classical viewpoint that the charge of the electron is a point but at the same time this point is never at rest and it is affected by the so called *zitterbewegung* and therefore it is moving in a confined region of size λ_C. This is the basic structure of the spinning particle models obtained in the kinematical formalism developed by the author [1] and also suggested by Dirac's analysis of the internal motion of the electron [2]. There, the charge of the particle is at a point $\boldsymbol{r}$, but this point is not the center of mass of the particle. In general the charge moves around the center of mass in a kind of harmonic or central motion. It is this motion of the charge that gives rise to the spin and dipole structure of the particle. In particular, the classical model that when quantised satisfies Dirac's equation shows, for the center of mass observer, a charge moving at the speed of light in circles of radius $R_o = \hbar/2mc$ and contained in a plane orthogonal to the spin direction [3]. It is this classical model of electron we shall consider in the subsequent analysis.

Therefore, to describe the dynamics of a charged spinning particle we have to follow the charge trajectory or, alternatively as we shall see later, the center of mass motion and the motion of the charge around the center of mass. In general the center of mass satisfies Newton-like dynamical equations in terms of the external force which is not defined at the center of mass position but rather at the position of the charge. The analysis of the radiation reaction is not considered here and is left to a future paper.

2 The invariant dynamical equation

Let us try to obtain the differential equation satisfied by the motion of a particular point of an arbitrary mechanical system. The method is to obtain first the whole family of trajectories of that point corresponding to all possible inertial observers. This family will be parameterised by some set of parameters and the elimination of these parameters among the function and their derivatives will do the job.

Then, let us consider the trajectory $\boldsymbol{r}(t)$, $t \in [t_1, t_2]$ followed by a point of a system for an arbitrary inertial observer. Any other inertial observer is related to the previous one by a transformation of a kinematical group such that their relative space-time measurements of any space-time event are given by

$$t' = T(t, \boldsymbol{r}; g_1, \ldots, g_r), \quad \boldsymbol{r}' = \boldsymbol{R}(t, \boldsymbol{r}; g_1, \ldots, g_r),$$

where the functions T and $\boldsymbol{R}$ define the action of the kinematical group G, of parameters $(g_1, \ldots, g_r)$, on space-time. Then the description of the trajectory of that point for observer O' is obtained from

$$t'(t) = T(t, \boldsymbol{r}(t); g_1, \ldots, g_r), \quad \boldsymbol{r}'(t) = \boldsymbol{R}(t, \boldsymbol{r}(t); g_1, \ldots, g_r), \quad \forall t \in [t_1, t_2].$$

If we eliminate t as a function of t' from the first equation and substitute into the second we shall get

$$\boldsymbol{r}'(t') = \boldsymbol{r}'(t'; g_1, \ldots, g_r). \tag{1}$$

Since observer O' is arbitrary, equation (1) represents the complete set of trajectories of the point for all inertial observers. Elimination of the r group parameters among the function $\boldsymbol{r}'(t')$ and their derivatives will give us the differential equation satisfied by the trajectory of the point. If G is either the Galilei or Poincaré group it is a ten-parameter group so that we have to work out in general up to the fourth derivative to obtain sufficient equations to eliminate the parameters. Therefore the order of the differential equation is dictated by the number of parameters and the structure of the kinematical group.

Let us analyse a simple example. We shall consider the free point particle. In the non-relativistic case the relationship of the inertial observer O with other inertial observer O' is given by the action of the Galilei group:

$$t' = t + b, \quad \boldsymbol{r}' = R(\boldsymbol{\alpha})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}, \tag{2}$$

where b and $\boldsymbol{a}$ represent the parameters of a time and space translation, respectively, $\boldsymbol{v}$ is the velocity of observer O as measured by O' and $R(\boldsymbol{\alpha})$ represents a rotation matrix that describes their relative orientation, expressed in terms of three parameters $\boldsymbol{\alpha}$ of a suitable parametrization of the rotation group.

For the free point particle it is possible to find a particular observer, the center of mass observer O^*, such that the trajectory of the particle for this observer reduces to

$$\boldsymbol{r}^*(t^*) \equiv 0, \quad \forall t^* \in [t_1^*, t_2^*].$$

and therefore its trajectory for any other observer O can be obtained from

$$t(t^*) = t^* + b, \quad \boldsymbol{r}(t^*) = \boldsymbol{v}t^* + \boldsymbol{a}. \tag{3}$$

In the relativistic case we have that the Poincaré group action is given by

$$t' = \gamma\left(t + \frac{\boldsymbol{v}\cdot R(\boldsymbol{\alpha})\boldsymbol{r}}{c^2}\right) + b, \tag{4}$$

$$\boldsymbol{r}' = R(\boldsymbol{\alpha})\boldsymbol{r} + \gamma\boldsymbol{v}t + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v}\cdot R(\boldsymbol{\alpha})\boldsymbol{r})\,\boldsymbol{v} + \boldsymbol{a}, \tag{5}$$

whith $\gamma = (1 - v^2/c^2)^{-1/2}$ and where the parameters have the same meaning and domains as in the non-relativistic case but here $v < c$. The trajectory of the point particle for observer O will be obtained from

$$t(t^*) = \gamma t^* + b, \quad \boldsymbol{r}(t^*) = \gamma\boldsymbol{v}t^* + \boldsymbol{a}. \tag{6}$$

Elimination of t^* in terms of t from the first equation of both (3) and (6) and substitution into the second yields the trajectory of the point for an arbitrary observer, which in the relativistic and non-relativistic formalism reduces to

$$\boldsymbol{r}(t) = (t - b)\boldsymbol{v} + \boldsymbol{a}.$$

Elimination of group parameters $\boldsymbol{v}$, b and $\boldsymbol{a}$ implies that the free evolution of a point particle satisfies the second order differential equation

$$\frac{d^2\boldsymbol{r}}{dt^2} = 0. \tag{7}$$

We must remark that equation (7) is independent of any group parameter and therefore its form is the same when written in any inertial reference frame. We thus obtain by this method an invariant dynamical equation.

But, what happens if the point we are analysing can never be find at rest during some finite time interval and for some particular inertial observer? This is the case of the position of the charge of the spinning particle models based upon the mentioned kinematical formalism [1].

3 The non-relativistic spinning electron

Let us assume as a model of electron the one in which for the center of mass observer the charge is moving in circles of radius R_o at the speed c in a plane orthogonal to the spin. We take this plane as the XOY plane in cartesian coordinates. We can use as natural space and time units R_o and R_o/c respectively, so that all space and time magnitudes are dimensionless and simmilarly the velocity, acceleration and further time derivatives. In these units the time observable for the center of mass observer is just the phase of this internal motion and the velocity of light takes the value $c = 1$.

For the center of mass observer, the trajectory of the charge of the electron is expressed in 3-vector form as

$$\boldsymbol{r}^*(t^*) = \begin{pmatrix} \cos t^* \\ \sin t^* \\ 0 \end{pmatrix}.$$

For the center of mass observer we get that

$$\frac{d^2\boldsymbol{r}^*(t^*)}{dt^{*2}} = -\boldsymbol{r}^*(t^*). \tag{8}$$

For any arbitrary inertial observer we get

$$\begin{aligned} t(t^*;g) &= t^* + b, \\ \boldsymbol{r}(t^*;g) &= R(\boldsymbol{\alpha})\boldsymbol{r}^*(t^*) + t^*\boldsymbol{v} + \boldsymbol{a}, \end{aligned}$$

where $g \equiv (b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ represents any arbitrary element of the Galilei group.

Because in this work we will be involved with time derivatives of order higher than two, instead of using as usual an over dot to represent them we shall define the order of derivation by an exponent enclosed in brackets as

$$\boldsymbol{r}^{(k)} \equiv \frac{d^k\boldsymbol{r}}{dt^k} = \frac{d}{dt^*}\left(\frac{d^{k-1}\boldsymbol{r}}{dt^{k-1}}\right)\frac{dt^*}{dt}.$$

In this non-relativistic case $dt^*/dt = 1$, then, after using (8) in some expressions we get the following derivatives

$$\begin{aligned} \boldsymbol{r}^{(1)} &= R(\boldsymbol{\alpha})\frac{d\boldsymbol{r}^*}{dt^*} + \boldsymbol{v}, \\ \boldsymbol{r}^{(2)} &= R(\boldsymbol{\alpha})\frac{d^2\boldsymbol{r}^*}{dt^{*2}} = -R(\boldsymbol{\alpha})\boldsymbol{r}^*, \\ \boldsymbol{r}^{(3)} &= -R(\boldsymbol{\alpha})\frac{d\boldsymbol{r}^*}{dt^*}, \\ \boldsymbol{r}^{(4)} &= -R(\boldsymbol{\alpha})\frac{d^2\boldsymbol{r}^*}{dt^{*2}} = R(\boldsymbol{\alpha})\boldsymbol{r}^* = -\boldsymbol{r}^{(2)}. \end{aligned}$$

Therefore, the differential equation satisfied by the position of the charge of a non-relativistic electron and for any arbitrary inertial observer is

$$\frac{d^4\boldsymbol{r}}{dt^4} + \frac{d^2\boldsymbol{r}}{dt^2} = 0. \tag{9}$$

The general solution of (9) is

$$\boldsymbol{r}(t) = \boldsymbol{A} + \boldsymbol{B}\,t + \boldsymbol{C}\cos t + \boldsymbol{D}\sin t,$$

which involves 12 arbitrary integration constants $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$. To select from this general solution the trajectories with circular motion some supplementary conditions must be imposed. The absolute value of the acceleration is Galilei invariant. Even more, $|\boldsymbol{r}^{(2)}|$ is constant for circular motions and this leads to $\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)}=0$, and making in (9) the scalar product with $\boldsymbol{r}^{(3)}$ we also get $\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(4)}=0$. These two conditions reduce to 10 the number of essential parameters of the general solution which imply $C=D$ and $\boldsymbol{C}\cdot\boldsymbol{D}=0$.

3.1 The center of mass

The center of mass position of the electron is defined as

$$\boldsymbol{q}=\boldsymbol{r}+\boldsymbol{r}^{(2)}, \tag{10}$$

because it reduces to $\boldsymbol{q}=0$ and $\boldsymbol{q}^{(1)}=0$ for the center of mass observer, so that dynamical equations can be rewritten in terms of the position of the charge and the center of mass as

$$\boldsymbol{q}^{(2)}=0, \quad \boldsymbol{r}^{(2)}=\boldsymbol{q}-\boldsymbol{r}. \tag{11}$$

Our fourth-order dynamical equation (9) can be split into two second order dynamical equations: A free equation for the center of mass and a central harmonic motion of the charge position $\boldsymbol{r}$ around the center of mass $\boldsymbol{q}$ of angular frequency 1 in these natural units.

In this non-relativistic case, since the transformation of the acceleration between inertial observers is given by

$$\boldsymbol{r}'^{(2)}=R(\boldsymbol{\alpha})\boldsymbol{r}^{(2)}.$$

It turns out that the definition of the center of mass position (10) implies by (2) that it transforms between inertial observers as

$$\boldsymbol{q}'=\boldsymbol{r}'+\boldsymbol{r}'^{(2)}=R(\boldsymbol{\alpha})\boldsymbol{q}+\boldsymbol{v}t+\boldsymbol{a},$$

The center of mass position for observer O', $\boldsymbol{q}'$, is just the Galilei transformed of $\boldsymbol{q}$. This is not true in general in the relativistic case.

3.2 Interaction with some external field

The free dynamical equation $\boldsymbol{q}^{(2)}=0$ is equivalent to $d\boldsymbol{P}/dt=0$, where $\boldsymbol{P}=m\boldsymbol{q}^{(1)}$ is the linear momentum of the system. Then our free equations should be replaced in the case of an interaction with an external electromagnetic field by

$$m\boldsymbol{q}^{(2)}=e[\boldsymbol{E}+\boldsymbol{r}^{(1)}\times\boldsymbol{B}], \quad \boldsymbol{r}^{(2)}=\boldsymbol{q}-\boldsymbol{r},$$

where in the Lorentz force the fields are defined at point $\boldsymbol{r}$ and it is the velocity of the charge that gives rise to the magnetic force term, while the second equation is left unchanged since it corresponds to the center of mass definition.

3.3 The spin dynamical equation

The relative position of the charge with respect to the center of mass is $\boldsymbol{k}=\boldsymbol{r}-\boldsymbol{q}$. The kinematical theory of spinning particles shows that the spin of this system reduces to the (anti)orbital angular momentum with respect to the center of mass of this relative position vector, *i.e*,

$$\boldsymbol{S}=-m\boldsymbol{k}\times\boldsymbol{k}^{(1)}. \tag{12}$$

The time derivative of this expression leads to

$$\frac{d\boldsymbol{S}}{dt} = -m\boldsymbol{k}\times\boldsymbol{k}^{(2)} = \boldsymbol{k}\times m\,\boldsymbol{q}^{(2)},$$

since $\boldsymbol{k}\times\boldsymbol{r}^{(2)} = 0$. The spin variation is equal to the torque, with respect to the center of mass, of the external Lorentz force applied at the point charge $\boldsymbol{r}$. If expressed in terms of the derivatives of the charge position, the spin takes the form

$$\boldsymbol{S} = m\,\boldsymbol{r}^{(3)}\times\boldsymbol{r}^{(2)}. \tag{13}$$

The spin is a constant of the motion for the free particle and thus the plane spanned by the relative vectors $\boldsymbol{k}$ and $\boldsymbol{k}^{(1)}$ (or $\boldsymbol{r}^{(2)}$ and $\boldsymbol{r}^{(3)}$) conserves its orientation. Because this plane contains the relative motion between both points we call this plane the zitterbewegung plane.

3.4 A work theorem

If we make the scalar product of $\boldsymbol{q}^{(1)}$ by the dynamical equation of $\boldsymbol{q}^{(2)}$, we have

$$m\boldsymbol{q}^{(1)}\cdot\boldsymbol{q}^{(2)} = e[\boldsymbol{E}+\boldsymbol{r}^{(1)}\times\boldsymbol{B}]\cdot\boldsymbol{q}^{(1)},$$

and thus

$$d\left(\frac{m}{2}(\boldsymbol{q}^{(1)})^2\right) = e[\boldsymbol{E}(t,\boldsymbol{r})+\boldsymbol{r}^{(1)}\times\boldsymbol{B}(t,\boldsymbol{r})]\cdot d\boldsymbol{q}.$$

The variation of the kinetic energy of the electron is the work done by the external Lorentz force, evaluated at the charge position, and along the center of mass trajectory.

4 The relativistic spinning electron

Let us assume the same electron model in the relativistic case. Since the charge is moving at the speed of light for the center of mass observer it is moving at this speed for every other inertial observer. Now, the relationship of space-time measurements between the center of mass observer and any arbitrary inertial observer is given by:

$$\begin{aligned} t(t^*;g) &= \gamma\left(t^* + \boldsymbol{v}\cdot R(\boldsymbol{\alpha})\boldsymbol{r}^*(t^*)\right) + b, \\ \boldsymbol{r}(t^*;g) &= R(\boldsymbol{\alpha})\boldsymbol{r}^*(t^*) + \gamma\boldsymbol{v}t^* + \frac{\gamma^2}{1+\gamma}\left(\boldsymbol{v}\cdot R(\boldsymbol{\alpha})\boldsymbol{r}^*(t^*)\right)\boldsymbol{v} + \boldsymbol{a}, \end{aligned}$$

where $g \equiv (b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ represents any arbitrary element of the Poincaré group.

If we define as before

$$\boldsymbol{r}^{(k)} = \frac{d^k\boldsymbol{r}}{dt^k} = \frac{d}{dt^*}\left(\frac{d^{k-1}\boldsymbol{r}}{dt^{k-1}}\right)\frac{dt^*}{dt},$$

but now with the shorthand notation for the following expressions:

$$\boldsymbol{K}(t^*) = R(\boldsymbol{\alpha})\boldsymbol{r}^*(t^*), \quad \boldsymbol{V}(t^*) = R(\boldsymbol{\alpha})\frac{d\boldsymbol{r}^*(t^*)}{dt^*} = \frac{d\boldsymbol{K}}{dt^*}, \quad \frac{d\boldsymbol{V}}{dt^*} = -\boldsymbol{K},$$

$$B(t^*) = \boldsymbol{v}\cdot\boldsymbol{K}, \quad A(t^*) = \boldsymbol{v}\cdot\boldsymbol{V} = \frac{dB}{dt^*}, \quad \frac{dA}{dt^*} = -B$$

where we have made use of equation (8) and

$$\frac{dt}{dt^*} = \gamma(1+A), \tag{14}$$

we obtain

$$\begin{aligned}
\boldsymbol{r}^{(1)} &= \frac{1}{\gamma(1+A)}\left(\boldsymbol{V} + \frac{\gamma}{1+\gamma}(1+\gamma+\gamma A)\boldsymbol{v}\right), &(15)\\
\boldsymbol{r}^{(2)} &= \frac{1}{\gamma^2(1+A)^3}\left(-(1+A)\boldsymbol{K} + B\boldsymbol{V} + \frac{\gamma}{1+\gamma}B\boldsymbol{v}\right), &(16)\\
\boldsymbol{r}^{(3)} &= \frac{1}{\gamma^3(1+A)^5}\left(-3B(1+A)\boldsymbol{K} - (1+A-3B^2)\boldsymbol{V} + \frac{\gamma}{1+\gamma}\left(A(1+A)+3B^2\right)\boldsymbol{v}\right) &(17)\\
\boldsymbol{r}^{(4)} &= \frac{1}{\gamma^4(1+A)^7}\Big((1+A)(1-2A-3A^2-15B^2)\boldsymbol{K} - \\
&\quad B(7+4A-3A^2-15B^2)\boldsymbol{V} - \frac{\gamma}{1+\gamma}(1-8A-9A^2-15B^2)B\boldsymbol{v}\Big). &(18)
\end{aligned}$$

From this we get

$$\begin{aligned}
\left(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(1)}\right)^2 &= 1, \quad \left(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(2)}\right) = 0, &(19)\\
\left(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)}\right) &= -\left(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(3)}\right) = \frac{1}{\gamma^4(1+A)^4}, &(20)\\
\left(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)}\right) &= -\frac{1}{3}\left(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(4)}\right) = \frac{2B}{\gamma^5(1+A)^6}, &(21)\\
\left(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}\right) &= \frac{1}{\gamma^6(1+A)^8}\left(1-A^2+3B^2\right), &(22)\\
\left(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(4)}\right) &= \frac{1}{\gamma^6(1+A)^8}\left(-1+2A+3A^2+9B^2\right), &(23)\\
\left(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(4)}\right) &= \frac{1}{\gamma^7(1+A)^{10}}\left(1+A+3B^2\right)4B. &(24)
\end{aligned}$$

From equations (20)-(22) we can express the magnitudes A, B and γ in terms of these scalar products between the different time derivatives $(\boldsymbol{r}^{(i)}\cdot\boldsymbol{r}^{(j)})$. The constraint that the velocity is 1 implies that all these and further scalar products for higher derivatives can be expressed in terms of only three of them. If the three equations (15)-(17) are solved in terms of the unknowns $\boldsymbol{v}$, $\boldsymbol{V}$ and $\boldsymbol{K}$ and substituded in (18), we obtain the differential equation satisfied by the charge position

$$\boldsymbol{r}^{(4)} - \frac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}\boldsymbol{r}^{(3)} + \left(\frac{2(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})} - \frac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^2} - (\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{1/2}\right)\boldsymbol{r}^{(2)} = 0. \tag{25}$$

It is a fourth order ordinary differential equation which contains as solutions only motions at the speed of light. In fact, if $(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(1)}) = 1$, then by derivation we have $(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(2)}) = 0$ and the next derivative leads to $(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)}) + (\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(3)}) = 0$. If we take this into account and make the scalar product of (25) with $\boldsymbol{r}^{(1)}$, we get $(\boldsymbol{r}^{(1)}\cdot\boldsymbol{r}^{(4)}) + 3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)}) = 0$, which is another relationship between the derivatives as a consequence of $|\boldsymbol{r}^{(1)}| = 1$.

For completion, the intermediate results are:

$$\begin{aligned}
\boldsymbol{v} &= \boldsymbol{r}^{(1)} - 3B\gamma(1+A)\boldsymbol{r}^{(2)} + \gamma^2(1+A)^3\boldsymbol{r}^{(3)}, &(26)\\
\boldsymbol{V} &= \frac{\gamma}{1+\gamma}\left(A\boldsymbol{r}^{(1)} + 3B\gamma(1+\gamma+\gamma A)(1+A)\boldsymbol{r}^{(2)} - \gamma^2(1+A)^3\left(1+\gamma+\gamma A\right)\boldsymbol{r}^{(3)}\right), &(27)\\
\boldsymbol{K} &= \frac{\gamma}{1+\gamma}\left(B\boldsymbol{r}^{(1)} + \gamma(1+A)[3B^2\gamma - (1+\gamma)(1+A)]\boldsymbol{r}^{(2)} - \gamma^3 B(1+A)^3\boldsymbol{r}^{(3)}\right). &(28)
\end{aligned}$$

and

$$\begin{aligned}
1+A &= \frac{8(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{5/2}}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{5/2} + 4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}) - 3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}, &(29)\\
B &= \frac{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{5/4}(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{5/2} + 4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}) - 3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}, &(30)\\
\gamma &= \frac{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{5/2} + 4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}) - 3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{8(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{11/4}}. &(31)
\end{aligned}$$

and therefore (23) and (24) are written

$$\begin{aligned}
(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(4)}) &= (\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2} + \frac{15(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})} - 2(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}), &(32)\\
(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(4)}) &= \left(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)}\right)\left[\frac{(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})} + \frac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^2} + (\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{1/2}\right]. &(33)
\end{aligned}$$

For the center of mass observer $\boldsymbol{r}^{*(i+2)} = -\boldsymbol{r}^{*(i)}$, $|\boldsymbol{r}^{*(i)}| = 1$, and thus $\boldsymbol{r}^{*(i)}\cdot\boldsymbol{r}^{*(i+1)} = 0$, for all i.

4.1 The center of mass

For the observer O^* the center of mass of the electron is at rest and located at the origin of its reference frame. What is the center of mass position for an arbitary inertial observer?

The velocity parameter $\boldsymbol{v}$ is the velocity of the observer O^* as measured by O. Although we do not know where the center of mass is, we take this value (26) as the velocity of the center of mass of the electron for the inertial observer O. Then

$$\boldsymbol{q}^{(1)} = \frac{d\boldsymbol{q}}{dt} \equiv \boldsymbol{v} = \boldsymbol{r}^{(1)} - \frac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})\,\boldsymbol{r}^{(2)} - 2(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})\,\boldsymbol{r}^{(3)}}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2} + (\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}) - \dfrac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}}. \tag{34}$$

We find by integration that the center of mass position is defined by

$$\boldsymbol{q} = \boldsymbol{r} + \frac{2(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})\,\boldsymbol{r}^{(2)}}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2} + (\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}) - \dfrac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}}. \tag{35}$$

We can check that both $\boldsymbol{q}$ and $\boldsymbol{q}^{(1)}$ vanish for the center of mass observer. The velocity $\boldsymbol{q}^{(1)}$ is in fact the time derivative of this expression and also a constant of the motion because its time derivative yields

$$\boldsymbol{q}^{(2)} = \frac{2(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2} + (\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}) - \dfrac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}}\times$$

$$\left[\boldsymbol{r}^{(4)} - \frac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}\boldsymbol{r}^{(3)} + \left(\frac{2(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})} - \frac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^2} - (\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{1/2}\right)\boldsymbol{r}^{(2)}\right],$$

where the factor between square brackets is the left hand side of expression (25) which vanishes. If we compute the following expressions

$$1-\boldsymbol{q}^{(1)}\cdot\boldsymbol{r}^{(1)} = \frac{2(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^2}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2} + (\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}) - \dfrac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}}, \tag{36}$$

$$(\boldsymbol{q}-\boldsymbol{r})^2 = \frac{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^3}{\left[(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2} + (\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)}) - \dfrac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}\right]^2}, \tag{37}$$

we see that the coefficient of the acceleration $\boldsymbol{r}^{(2)}$ in (35) is $(\boldsymbol{q}-\boldsymbol{r})^2/(1-\boldsymbol{q}^{(1)}\cdot\boldsymbol{r}^{(1)})$, and it turns out that the fourth order dynamical equation for the position of the charge (25) can be rewritten as a system of two second order differential equations for the positions $\boldsymbol{q}$ and $\boldsymbol{r}$

$$\boldsymbol{q}^{(2)} = 0, \quad \boldsymbol{r}^{(2)} = \frac{1-\boldsymbol{q}^{(1)}\cdot\boldsymbol{r}^{(1)}}{(\boldsymbol{q}-\boldsymbol{r})^2}(\boldsymbol{q}-\boldsymbol{r}). \tag{38}$$

Simmilarly as for the non-relativistic electron we get a free motion for the center of mass and a central motion around $\boldsymbol{q}$ for the position of the charge. The second equation for $\boldsymbol{r}^{(2)}$ is just a new form of writing the center of mass definition (35). These equations are relativistic invariant because are written in the same form in every inertial frame. If $\boldsymbol{r}'$ is the position of the charge for the observer O', it can be expressed in terms of $\boldsymbol{r}$ by the transformation equations (4). However the center of mass position $\boldsymbol{q}'$ is not the corresponding transformed of $\boldsymbol{q}$, but it is expressed in O' frame in terms of $\boldsymbol{r}'$ and its derivatives by (35). The center of mass position does not transform like the spatial component of a four-vector. Nevertheless, the four-velocity associated to the motion of the center of mass $v^\mu \equiv (\gamma(q^{(1)}), \gamma(q^{(1)})\boldsymbol{q}^{(1)})$ transforms like a true four-vector.

4.2 Interaction with some external field

The free equation for the center of mass motion $\boldsymbol{q}^{(2)} = 0$, represents the conservation of the linear momentum $d\boldsymbol{P}/dt = 0$. But the linear momentum is written in terms of the center of mass velocity as $\boldsymbol{P} = m\gamma(q^{(1)})\boldsymbol{q}^{(1)}$, so that the free dynamical equations (38) in the presence of an external field should be replaced by

$$\boldsymbol{P}^{(1)} = \boldsymbol{F}, \quad \boldsymbol{r}^{(2)} = \frac{1-\boldsymbol{q}^{(1)}\cdot\boldsymbol{r}^{(1)}}{(\boldsymbol{q}-\boldsymbol{r})^2}(\boldsymbol{q}-\boldsymbol{r}), \tag{39}$$

where $\boldsymbol{F}$ is the external force and the second equation is left unchanged because we consider, even with interaction, the same definition of the center of mass position. If the external force is produced by some external electromagnetic field, then

$$\boldsymbol{F} = e\left[\boldsymbol{E} + \boldsymbol{r}^{(1)}\times\boldsymbol{B}\right],$$

where the fields $\boldsymbol{E}(t,\boldsymbol{r})$ and $\boldsymbol{B}(t,\boldsymbol{r})$ are defined at the position of the charge and where it is the velocity of the charge that produces the magnetic force term. If we express the derivative of the linear momentum in terms of the center of mass velocity and acceleration

$$\frac{d\boldsymbol{P}}{dt} = m\gamma(q^{(1)})\boldsymbol{q}^{(2)} + m\gamma(q^{(1)})^3(\boldsymbol{q}^{(1)}\cdot\boldsymbol{q}^{(2)})\boldsymbol{q}^{(1)}$$

we get

$$m\gamma(q^{(1)})^3(\boldsymbol{q}^{(1)}\cdot\boldsymbol{q}^{(2)})=\boldsymbol{F}\cdot\boldsymbol{q}^{(1)}$$

and by leaving the highest derivative $\boldsymbol{q}^{(2)}$ on the left hand side we finally get the differential equations which describe the evolution of a relativistic spinning electron in the presence of an external electromagnetic field:

$$m\boldsymbol{q}^{(2)} = \frac{e}{\gamma(q^{(1)})}\left[\boldsymbol{E}+\boldsymbol{r}^{(1)}\times\boldsymbol{B}-\boldsymbol{q}^{(1)}\left(\left[\boldsymbol{E}+\boldsymbol{r}^{(1)}\times\boldsymbol{B}\right]\cdot\boldsymbol{q}^{(1)}\right)\right], \tag{40}$$

$$\boldsymbol{r}^{(2)} = \frac{1-\boldsymbol{q}^{(1)}\cdot\boldsymbol{r}^{(1)}}{(\boldsymbol{q}-\boldsymbol{r})^2}(\boldsymbol{q}-\boldsymbol{r}). \tag{41}$$

4.3 The spin dynamical equation

If we call again $\boldsymbol{k}=\boldsymbol{r}-\boldsymbol{q}$ to the relative position of the charge with respect to the center of mass, the spin of the system is the (anti)orbital angular momentum of the charge motion around the center of mass,

$$\boldsymbol{S}=-m\boldsymbol{k}\times\boldsymbol{k}^{(1)}. \tag{42}$$

This is consistent with the usual torque equation because the time variation of the spin is

$$\frac{d\boldsymbol{S}}{dt}=-m\boldsymbol{k}\times\boldsymbol{k}^{(2)}=\boldsymbol{k}\times m\,\boldsymbol{q}^{(2)},$$

because $\boldsymbol{k}\times\boldsymbol{r}^{(2)}=0$ and which represents the torque, with respect to the center of mass, of the right hand side of equation (40) considered as a force applied at the point charge $\boldsymbol{r}$. The explicit form of the spin in terms of the derivatives of the charge position is,

$$\boldsymbol{S}=\frac{4m(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^2}{\left[(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2}+(\boldsymbol{r}^{(3)}\cdot\boldsymbol{r}^{(3)})-\dfrac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}\right]^2}\,\boldsymbol{r}^{(3)}\times\boldsymbol{r}^{(2)}. \tag{43}$$

As in the non-relativistic case we call the plane spanned by $\boldsymbol{r}^{(2)}$ and $\boldsymbol{r}^{(3)}$, the zitterbewegung plane.

4.4 Other constants of the motion

We have found that the spin (42) is a constant of the motion for a free particle. We can obtain in the free evolution other constants of the motion. If we define the vector magnitude

$$\boldsymbol{n}=\frac{\boldsymbol{r}^{(2)}\times\boldsymbol{r}^{(3)}}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2}}, \tag{44}$$

in the spin direction and which is orthogonal to the zitterbewegung plane, then $\boldsymbol{n}^{(1)}=0$. In fact, the derivative of (44) leads to

$$\frac{1}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})^{3/2}}\left[\boldsymbol{r}^{(2)}\times\boldsymbol{r}^{(4)}-\frac{3(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)}\cdot\boldsymbol{r}^{(2)})}\,\boldsymbol{r}^{(2)}\times\boldsymbol{r}^{(3)}\right],$$

where the term between square brackets is just the vector or cross product of the left hand side of (25) by $\boldsymbol{r}^{(2)}$ from the left. Since $\boldsymbol{n}$ is a conserved vector its direction is also a conserved magnitude and therefore the plane spanned by $\boldsymbol{r}^{(2)}$ and $\boldsymbol{r}^{(3)}$ conserves its orientation in space during the free motion. This implies that the unit vector $\widehat{\boldsymbol{n}}=\boldsymbol{n}/|\boldsymbol{n}|$ is also another constant of the motion as can be checked.

4.5 A work theorem

If we make the scalar product of both sides of equation (40) by $\boldsymbol{q}^{(1)}$ we get

$$m\boldsymbol{q}^{(1)} \cdot \boldsymbol{q}^{(2)} = \frac{e}{\gamma(q^{(1)})^3}\left[\boldsymbol{E} + \boldsymbol{r}^{(1)} \times \boldsymbol{B}\right] \cdot \boldsymbol{q}^{(1)},$$

i.e.,

$$d\left(m\gamma(q^{(1)})\right) = e\left[\boldsymbol{E} + \boldsymbol{r}^{(1)} \times \boldsymbol{B}\right] \cdot d\boldsymbol{q},$$

so that the variation of the energy of the electron is the work done by the external Lorentz force along the center of mass trajectory.

We must remark again that the fields are defined at the position of the charge. In smooth fields of very small local variation and for a low center of mass velocity the average value of $\boldsymbol{r}^{(1)}$ during a complete turn of the electron charge is basically $\boldsymbol{q}^{(1)}$ and it turns out that the magnetic field produces no work on the electron. But nevertheless this formalism suggests many other situations in which local variations of the fields or equivalently non negligible field gradients at distances of order of λ_C may produce local changes of the energy of the particle. This might be the case of the intersticial magnetic field of ferromagnetic materials, for instance, as well as in diverse condensed matter physics phenomena. One of these features was considered as giving rise to a plausible explanation of the magnetoresistence of polycrystaline films by spin polarised tunneling [4].

Acknowledgments

This work was supported by The University of the Basque Country under the project UPV/EHU 172.310 EB 150/98 and General Research Grant UPV 172.310-G02/99.

References

[1] M. Rivas, J. Phys. A, **18** (1985) 1971; J. Math. Phys., **30** (1989) 318. For a more detailed exposition see M. Rivas, *Kinematical theory of spinning particles*, (Kluwer, Dordrecht 2001).

[2] P.A.M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Oxford U. P. 1958), §69.

[3] M. Rivas, J. Math. Phys., **35** (1994) 3380.

[4] M. Rivas, Phys. Lett. **A 248** (1998) 279.